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The holographic ward identity: examples from 2+1 gravity

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Abstract: In the AdS/CFT correspondence the boundary Ward identities are encoded in the bulk constraints. We study the three-dimensional version of this result using the Chern-Simons formulation of gravity. Due to the metric boundary conditions the conformal identities cannot be derived in a straightforward way from the chiral ones. We pay special attention to this case and find the necessary modifications to the chiral currents in order to find the two Virasoro operators. The supersymmetric Ward identities are studied as well.

Keywords: AdS-CFT and dS-CFT Correspondence, Models of Quantum Gravity, 2D Gravity.

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Contents

1. Introduction

One of the most studied examples of the AdS/CFT correspondence is the relation between three-dimensional gravity with a negative cosmological constant and two-dimensional Conformal Field Theory. Even before the AdS/CFT was discovered, Brown and Henneaux [1] found a deep connection between 3d gravity with anti-de Sitter boundary conditions and the 2d centrally-extended conformal algebra. This connection was then reformulated in [2] as a relationship between three dimensional gravity and Liouville theory (and generalized in [3] to supergravity). Within the AdS/CFT correspondence one can understand this result as a relationship between fluctuations on the AdS background (the Liouville field) and vacuum expectation values of the 2d CFT.

It is unfortunate that quantum Liouville theory is not fully understood. There are questions one can ask at the gravity side which do not yet have an answer within Liouville theory, e.g., the black hole microstates (see, for example, [4]). Whether or not Liouville theory encodes the necessary information to answer all questions related to 3d dimensional quantum gravity is still not known. See [5] for a recent discussion on Liouville theory.

The relationship between 3d gravity and Liouville theory discovered in [2] is based on the fact that three-dimensional gravity does not have any local degrees of freedom. Using the relationship between Chern-Simons and WZW theories [6, 7] together with the hamiltonian reduction technics [8], it was shown in [2] that the (local) relevant dynamical information is encoded in the Liouville action. From a hamiltonian point of view, the Liouville field describes those "would-be-trivial" degrees of freedom associated to diffeomorphisms which preserve the boundary conditions, but have non-zero generators [9].

On the other hand, in the general AdS/CFT correspondence [10]–[12] the CFT correlators are computed directly from a classical action on the AdS background. In particular, if one is interested in correlators involving the energy-momentum tensor, the needed action is precisely the (renormalised) gravity action $I_{ren}[g_{ij}] \equiv W[g_{ij}]$ as a function of an *arbitrary* induced boundary metric g_{ij} which acts as a source in the CFT. Performing a near boundary analysis the needed divergences and counterterms can be identified, and $W[g_{ij}]$ can be constructed explicitely [13].

The AdS/CFT correspondence establishes also a relationship between the CFT Ward identities and RG equations with the bulk equations. The radial ("evolution") equation is associated to the CFT renormalization equation, while the constraints to Ward identities [14]–[16]. This relationship becomes particularly nice and clear in three-dimensional Chern-Simons gravity. In fact, it has been known for a long time [17]–[19] that a two-dimensional flat $\mathrm{SL}(2,\mathrm{Re})$ current encodes the Virasoro Ward identity, and this result fits quite naturally within AdS/CFT. In this paper we shall put these results in the AdS/CFT context. Just as a single $\mathrm{SL}(2,\mathrm{Re})$ field contains the Virasoro Ward identity, the $\mathrm{SL}(2,\mathrm{Re}) \times \mathrm{SL}(2,\mathrm{Re})$ theory contains the conformal identity. However, the construction is slightly more complicated because the boundary conditions couple the left and right gauge fields. We shall see that the AdS/CFT prescription provides a natural reason to decouple both sectors leading to the conformal identities. We shall also see that the AdS/CFT classification of the gauge field components (leading, first subleading and subleading) provides a natural, and free of ad-hoc assumptions, derivation of these identities from the bulk constraints.

The paper is organized as follows. After a brief introduction to the Virasoro Ward identity and Polyakov's action, in section 2 we analyze the bosonic conformal Ward identity, derived from AdS_3/CFT_2. In section 3 we consider the $\mathrm{Osp}(2,1) \times \mathrm{Osp}(2,1)$ theory and find the superconformal Ward identities. Finally, we also consider the extended models in the chiral case.

2. Bosonic AdS$_3$/CFT$_2$

2.1 The Virasoro ward identity

Let us start by recalling that in bosonic case the generator of energy-momentum tensor correlators, $W[g]$, is given by Polyakov's non-local action

$$W[g] = \frac{c}{24\pi} \int\!\!\int R \frac{1}{\nabla^2} R \,, \tag{2.1}$$

where c is the central charge. The derivation of this functional from AdS/CFT was studied in [20] (see also [21]). In Polyakov's light cone gauge

$$ds^2 = dzd\bar{z} + h(z,\bar{z})d\bar{z}^2 \,. \tag{2.2}$$

the metric has only one independent function $h(z,\bar{z})$ and one defines $T = \delta W/\delta h$. Then, directly from the trace anomaly and conservation of T^{ij} it follows that T satisfies [18],

$$\bar{\partial} T = -\frac{c}{12}\partial^3 h + 2\partial h T + h\partial T \tag{2.3}$$

which is known as the Virasoro Ward identity.

This relation is contained in a flat SL(2, Re) 2d gauge field as follows [17]. Consider in two dimensions the gauge field $A = A_z dz + A_{\bar{z}} d\bar{z}$ with the following particular values

$$A_z = \begin{pmatrix} 0 & 1 \\ T/k & 0 \end{pmatrix}, \qquad A_{\bar{z}} = \begin{pmatrix} \alpha & \mu \\ \beta & -\alpha \end{pmatrix}. \tag{2.4}$$

All functions T, α, β, μ are functions of z and $\bar{z}$. The flatness condition $F_{z\bar{z}} = \partial A_{\bar{z}} - \bar{\partial} A_z + [A_z, A_{\bar{z}}] = 0$ implies

$$\alpha = \frac{1}{2}\partial\mu \tag{2.5}$$

$$\beta = \frac{1}{k}\,\mu T - \frac{1}{2}\partial^2\mu \tag{2.6}$$

$$\bar{\partial} T = \mu\partial T + 2\partial\mu\, T - \frac{k}{2}\partial^3\mu\,. \tag{2.7}$$

We recognize the third equation (2.7) as the Virasoro Ward identity with $c = 6k$.

Since three-dimensional gravity can be formulated as the difference of SL(2, Re) Chern-Simons actions[22] it is hardly surprising that, within AdS/CFT, the Virasoro Ward identities will arise through this mechanism. Actually, the relationship between SL(2, Re) Chern-Simons theory (from where the constraint $F_{z\bar{z}} = 0$ can be derived) and the 2-dimensional action (2.1) was studied in detail in [19], almost ten years before the AdS/CFT correspondence became available. A similar idea was applied to W-gravity in [23]. Our goal in this paper is to put some of the results of [19] in the AdS/CFT perspective, clarifying some issues, for example how to deal with the second SL(2, Re) field, and to extend them to supergravity.

We shall see that the AdS/CFT correspondence provides a natural interpretation on why the above mechanism works. In particular we shall see that μ enters as the source associated to T, α is a subleading term related locally to the source, and β, T are non-local, as in (2.7). We shall also extend this mechanism to the non-chiral case, which is not trivial due to the boundary conditions on the metric.

2.2 Classical three-dimensional equations and the conformal ward identity

We start by writing the solution to the three-dimensional gravitational equations in the Fefferham-Graham (FG) form. In this paragraph we follow closely [21].

Let A and $\bar{A}$ be two SL(2, Re) fields related to the triad and spin connection in the usual way

$$A = w + e\,, \qquad \bar{A} = w - e\,. \tag{2.8}$$

The solution to the equations of motion can be locally written as

$$A = g^{-1}dg\,, \qquad \bar{A} = h^{-1}dh \tag{2.9}$$

for arbitrary group elements g and h. Since (2.9) is the most general local solution to the equations of motion we should be able to find the Fefferham-Graham (FG) [25] expansion for the metric by manipulating (2.9). Incidentally, we recall that, in three dimensions, the

FG expansion is finite and has only three-terms [20] (see [24] for a different formulation of this fact).

The FG expansion makes use of a choice of coordinates, namely, $g_{ri} = 0$. Our first task is then to write the solutions to the equations of motion (2.9) in a way consistent with the FG expansion, and coordinate choice. As shown in [21] the FG expansion is implemented in the Chern-Simons formulation by writting the fields in the form,

$$\begin{aligned} A(x,r) &= e^{-rJ_3}\, A(x)\, e^{rJ_3} + dr\, J_3 && (2.10)\\ \bar{A}(x,r) &= e^{rJ_3}\, \bar{A}(x)\, e^{-rJ_3} - dr\, J_3\,. && (2.11)\end{aligned}$$

where $A(x) = A_i(x)dx^i$ and $\bar{A}(x) = \bar{A}_i(x)dx^i$ are both SL(2, Re)−valued connections defined on the surface at fixed r. The radial dependence is then explicitly factor out as a gauge transformation. The fields $A(x,r)$ and $\bar{A}(x,r)$ will satisfy the equations of motion provided $A(x)$ and $\bar{A}(x)$ are both *flat*.

The fields $A(x)$ and $\bar{A}(x)$ are in the SL(2, Re) algebra and we can expand them in the basis J_+, J_-, J_3, namely,

$$A(x) = A^+(x)J_+ + A^-(x)J_- + A^3(x)J_3 \qquad (2.12)$$

and similarly for the other copy. Replacing these expansions in (2.10,2.11), and using $e^{-rJ_3}J_\pm e^{rJ_3} = e^{\mp r}\, J_\pm$, we obtain,

$$\begin{aligned} A(x,r) &= (dr + A^3(x))J_3 + e^{-r}\, A^+(x)J_+ + e^{r}\, A^-(x)J_-\,, && (2.13)\\ \bar{A}(x,r) &= (-dr + \bar{A}^3(x))J_3 + e^{r}\, \bar{A}^+(x)J_+ + e^{-r}\, \bar{A}^-(x)J_-\,. && (2.14)\end{aligned}$$

We now see that the different components of $A(x)$ and $\bar{A}(x)$ appears multiplied by different powers of e^r. In the limit $r \to \infty$, this expansion assigns A^- the role of leading term ("source"), A^3 the next to leading term, and A^+ as the subleading term, which will be non-local in the source. The expansion has only three terms as anticipated previously [20]. A similar analysis holds for $\bar{A}$.

Let us now compute the corresponding three-dimensional triad [21]

$$e = \left(dr + \frac{1}{2}(A^3 - \bar{A}^3)\right) J_3 + \frac{1}{2}e^{r}(A^-J_- - \bar{A}^+J_+) + \frac{1}{2}e^{-r}(A^+J_+ - \bar{A}^-J_-)\,. \qquad (2.15)$$

Since $A(x)$ and $\bar{A}(x)$ are forms on the 2-surface, this vielbein does not have the FG form. In fact the metric $g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab}$ will have a piece of the form $dr(A^3_i - \bar{A}^3_i)dx^i$. In order to conform with the FG choice of coordinates we need to impose the extra boundary condition [21]

$$A^3(x) = \bar{A}^3(x)\,. \qquad (2.16)$$

This condition turns out to be the key input in order to find the Virasoro Ward identities, including the supersymmetric ones. In particular, we stress that it is *not* correct to impose $A^3_z = 0$ and $\bar{A}^3_{\bar{z}} = 0$ separately, as one would naively conclude from (2.4). Only their difference has to be zero, and this condition follows in a natural way from the FG choice of coordinates.

Summarizing, we have two SL(2, Re) connections A and $\bar{A}$ defined on the surface at fixed r. They are completely arbitrary except for the condition (2.16). Now, the different components of A and $\bar{A}$ play different roles within the AdS/CFT prescription. The leading components, A^- and $\bar{A}^+$, are fixed and represents the sources for the corresponding CFT operator, analogous to $g_{(0)ij}$ in the metric FG expansion. The other components are expressed in terms of the fixed components A^- and $\bar{A}^+$ via the equations of motion. Some of these relations are local, and others non-local. The 1-point functions will be related non-locally to A^- and $\bar{A}^+$.

The sources $A^-, \bar{A}^+$ are arbitrary and we can consider particular values for it, if we only want to turn on some particular operators. For example, Polyakov's light cone gauge (2.2) leaves enough information to find one copy of the Virasoro algebra. We would like to consider here a more general situation in which both copies of the Virasoro algebra arise simultaneously. To this end, we choose complex coordinates $z, \bar{z}$ and consider boundary metrics having two Beltrami parameters,

$$ds^2_{(0)} = |d\bar{z} + \mu(z,\bar{z})dz|^2 \,. \tag{2.17}$$

The appropriated bulk gauge field that has as leading term the metric (2.17), and fulfills (2.16), is

$$A_z(z,\bar{z}) = \begin{pmatrix} \omega/2 & T'/k \\ 1 & -\omega/2 \end{pmatrix} \qquad \overline{A}_z(z,\bar{z}) = \begin{pmatrix} \omega/2 & \bar{\mu} \\ \bar{\gamma} & -\omega/2 \end{pmatrix} \tag{2.18}$$

$$A_{\bar{z}}(z,\bar{z}) = \begin{pmatrix} \bar{\omega}/2 & \gamma \\ \mu & -\bar{\omega}/2 \end{pmatrix} \qquad \overline{A}_{\bar{z}}(z,\bar{z}) = \begin{pmatrix} \bar{\omega}/2 & 1 \\ \bar{T}'/k & -\bar{\omega}/2 \end{pmatrix} \tag{2.19}$$

Here all functions depend on $z, \bar{z}$. μ and $\bar{\mu}$ are the sources; they enter as the leading terms when (2.18), (2.19) are replaced in the expansion (2.10), (2.11). All other components become functions of μ and $\bar{\mu}$ via the equations of motion.

The bulk constraint equations $F = \partial A_{\bar{z}} - \overline{\partial} A_z + [A_z, A_{\bar{z}}] = 0$ (and similarly for $\bar{F} = 0$) imply the following relations, which can be understood as the non-chiral version of (2.7),

$$\omega = \bar{\omega}\bar{\mu} + \overline{\partial}\bar{\mu} \tag{2.20}$$

$$\bar{\omega} = \omega\mu - \partial\mu \tag{2.21}$$

$$\gamma = \frac{1}{2}(\partial\bar{\omega} - \overline{\partial}\omega) + \frac{\mu T'}{k} \tag{2.22}$$

$$\bar{\gamma} = \frac{1}{2}(\partial\bar{\omega} - \overline{\partial}\omega) + \frac{\mu\bar{T}'}{k} \tag{2.23}$$

$$\overline{\partial} T'/k = \partial\gamma + \omega\gamma - \frac{\bar{\omega}T'}{k} \tag{2.24}$$

$$\partial\bar{T}'/k = \overline{\partial}\delta - \bar{\omega}\delta + \frac{\omega\bar{T}'}{k} \,. \tag{2.25}$$

As anticipated previously, ω and $\bar{\omega}$ can be expressed locally in terms of the sources μ and $\bar{\mu}$ by solving (2.20), (2.21). Incidentally, we mention that the 1-form $w = \omega dz + \bar{\omega} d\overline{z}$ turns out to be the spin connection associated to the metric (2.17), equations (2.20)–(2.21) are the corresponding torsion free conditions, and

$$R = dw \tag{2.26}$$

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becomes the two-dimensional curvature.

The functions γ and $\bar{\gamma}$ can also be eliminated algebraically using (2.22), (2.23), although they depend on T' and $\bar{T}'$ which are non-local in the sources.

Finally, by analogy with (2.7), one would expect Equations (2.24,2.25) to give two copies of the Virasoro Ward identity. To see this, however, one needs to add the following *local* pieces to T' and $\bar{T}'$ [19],

$$T = T' + \frac{k}{2}\left(\frac{1}{2}\omega^2 + \partial\omega\right), \tag{2.27}$$

$$\overline{T} = \overline{T}' + \frac{k}{2}\left(\frac{1}{2}\overline{\omega}^2 - \overline{\partial}\overline{\omega}\right). \tag{2.28}$$

It is direct to see that T and $\bar{T}$ satisfy the correct Ward identities

$$\overline{\partial}T = -\frac{k}{2}\partial^3\mu + 2(\partial\mu)T + \mu\partial T \tag{2.29}$$

$$\partial\overline{T} = -\frac{k}{2}\overline{\partial}^3\overline{\mu} + 2(\overline{\partial}\overline{\mu})\overline{T} + \overline{\mu}\overline{\partial}\overline{T}. \tag{2.30}$$

Summarizing, we have shown that the constraint equations of three-dimensional gravity contain the conformal Ward identity. It remains to prove that the combinations T and $\bar{T}$ represent the 1-point functions in the presence of the sources μ and $\bar{\mu}$.

2.3 The action and 1-point vacuum expectation values

In order to prove that T and $\bar{T}$ are the correct 1-point functions we need to consider the renormalised 3d action written as a function of the sources μ and $\bar{\mu}$, $W[\mu,\bar{\mu}]$, and check that

$$T = -2\pi\frac{\delta W[\mu,\bar{\mu}]}{\delta\mu}, \qquad \bar{T} = -2\pi\frac{\delta W[\mu,\bar{\mu}]}{\delta\bar{\mu}}. \tag{2.31}$$

Three-dimensional gravity is given by the difference of two Chern-Simons actions $I[A,\bar{A}] = I[A] - I[\bar{A}]$ with $I[A] = \frac{k}{4\pi}\int\left(AdA + \frac{2}{3}AAA\right)$. We then start by looking at the on-shell variation of $I[A,\bar{A}]$,

$$\delta I[A,\bar{A}] = -\frac{k}{4\pi}\int_{\partial M}(A\delta A - \bar{A}\delta\bar{A}) \tag{2.32}$$

(Here we have omitted the terms that vanishes when the equations of motion hold.) Replacing here the asymptotic form of the gauge field (2.10,2.11) and (2.22), (2.23) we obtain the *finite* result

$$\delta I[A,\bar{A}] = -\frac{1}{2\pi}\int_{\partial M}\left(T'\delta\mu + \bar{T}'\delta\bar{\mu} + \frac{k}{2}\delta R\right), \tag{2.33}$$

where $R = dw$ is the two-dimensional curvature (see (2.26)). This is almost what we need. R is a local function of the sources that can be added (with the opposite sign) to $I[A,\bar{A}]$, as dictated by the Dirichlet problem with μ and $\bar{\mu}$ fixed. However, recall that the true 1-point functions satisfying the correct Ward identities are $T, \bar{T}$, defined in (2.27,2.28), and

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not $T', \bar{T}'$. However, it follows that the term which transform $(T', \bar{T}')$ into $(T, \bar{T})$ in (2.33) is also a trivial local term. To see this consider the following identity

$$0 = (\omega\delta\overline{\omega} - \overline{\omega}\delta\omega) - (\omega\delta\overline{\omega} - \overline{\omega}\delta\omega) \tag{2.34}$$

$$= \left(\frac{1}{2}\omega^2 + \partial\omega\right)\delta\mu + \left(\frac{1}{2}\overline{\omega}^2 - \overline{\partial}\overline{\omega}\right)\delta\overline{\mu} + \frac{1}{2}\delta\left(\partial\mu\omega - \overline{\partial}\overline{\mu}\overline{\omega}\right). \tag{2.35}$$

To go from (2.34) to (2.35) we have used (2.21) and (2.20) in the first and second terms in (2.34), respectively. We then observe that the first two terms in (2.35) are exactly the local pieces added to $(T', \bar{T}')$ in (2.27), (2.28), multiplied by the variations of the sources. We then find that the sum $(\frac{1}{2}\omega^2 + \partial\omega)\delta\mu + (\frac{1}{2}\overline{\omega}^2 - \overline{\partial}\overline{\omega})\delta\overline{\mu}$ is equal to the total variation of a local term that we call $K[\mu, \bar{\mu}] = \int_{\partial M}(\partial\mu w_z - \bar{\partial}\bar{\mu}w_{\bar{z}})$, and hence trivial. Note also that $K[\mu, \bar{\mu}]$ is precisely the interaction term, found in [19], which decouples in the Polyakov action the actions for μ and $\bar{\mu}$.

In summary, the Dirichlet problem instruct us to consider the improved action (up to divergent counterterms which do not contribute to the 1-point function) $W[\mu, \bar{\mu}] := I[A, \bar{A}] + \frac{k}{4\pi}\int_{\partial M}(R + \frac{1}{2}K)$ whose variation is given by,

$$\delta W[\mu, \bar{\mu}] = -\frac{1}{2\pi}\int_{\partial M}\left(T\delta\mu + \bar{T}\delta\bar{\mu}\right), \tag{2.36}$$

in full agreement with (2.31), as desired.

3. Supergravity and the superconformal ward identity

3.1 Holographic super Virasoro ward identities

We will now generalize the results obtained in the previous section to the super-symmetric case. The discussion is generalized to this case with almost no changes. We then only briefly state the form of the currents and their Ward identities.

Three dimensional super-gravity can be recast as the difference of two Chern-Simons action for the OSp(1|2) group. Let $\Gamma = A^a J_a + \chi^{\pm}R_{\pm}$ and $\bar{\Gamma} = \bar{A}^a J_a + \bar{\chi}^{\pm}R_{\pm}$ be two OSp(1|2) connections. The radial coordinate dependence of the connections is given by (see (2.10,2.11)):

$$\begin{aligned}\Gamma(x,r) &= (dr + A^3)J_3 + e^{-r}A^+J_+ + e^rA^-J_- + e^{-r/2}\chi^+R_+ + e^{r/2}\chi^-R_- \\ \bar{\Gamma}(x,r) &= (-dr + \bar{A}^3)J_3 + e^r\bar{A}^+J_+ + e^{-r}\bar{A}^-J_- + e^{r/2}\bar{\chi}^+R_+ + e^{-r/2}\bar{\chi}^-R_-\end{aligned} \tag{3.1}$$

Imposing the condition (2.16) and using (3.1) to identify the leading terms, we set the gauge fields to be:

$$\Gamma_z(x) = \begin{pmatrix} \omega/2 & T'/k & Q/k \\ 1 & -\omega/2 & 0 \\ 0 & Q/k & 0 \end{pmatrix} \qquad \bar{\Gamma}_z(x) = \begin{pmatrix} \omega/2 & 2\pi\bar{\mu} & 2\pi\bar{\epsilon} \\ \delta & -\omega/2 & \theta \\ -\theta & 2\pi\bar{\epsilon} & 0 \end{pmatrix} \tag{3.2}$$

$$\Gamma_{\bar{z}}(x) = \begin{pmatrix} \bar{\omega}/2 & \gamma & \psi \\ 2\pi\mu & -\bar{\omega}/2 & 2\pi\epsilon \\ -2\pi\epsilon & \psi & 0 \end{pmatrix} \qquad \bar{\Gamma}_{\bar{z}}(x) = \begin{pmatrix} \bar{\omega}/2 & 1 & 0 \\ \bar{T}'/k & -\bar{\omega}/2 & \bar{Q}/k \\ -\bar{Q}/k & 0 & 0 \end{pmatrix}, \tag{3.3}$$

where $(\mu, \bar{\mu}, \epsilon, \bar{\epsilon})$ are the sources and the induced 2-d boundary metric is given by (2.17).

Solving for a flat connection, $F = \partial\Gamma_{\bar{z}} - \bar{\partial}\Gamma_z + [\Gamma_z, \Gamma_{\bar{z}}] = 0$ (and $\bar{F} = 0$), we get the super-Virasoro Ward identities:

$$\begin{aligned}
\frac{1}{2\pi}\bar{\partial}T &= -\frac{k}{2}\partial^3\mu + 2(\partial\mu)T + \mu\partial T + \partial(\epsilon Q) + 2(\partial\epsilon)Q && (3.4)\\
\frac{1}{2\pi}\bar{\partial}Q &= -k\partial^2\epsilon + \epsilon T + \partial(\mu Q) + \frac{1}{2}\partial\mu Q && (3.5)\\
\frac{1}{2\pi}\partial\bar{T} &= -\frac{k}{2}\bar{\partial}^3\bar{\mu} + 2(\bar{\partial}\bar{\mu})\overline{T} + \bar{\mu}\bar{\partial}\bar{T} + \bar{\partial}(\bar{Q}\bar{\epsilon}) + 2\bar{Q}(\bar{\partial}\bar{\epsilon}) && (3.6)\\
\frac{1}{2\pi}\partial\bar{Q} &= -k\bar{\partial}^2\bar{\epsilon} + \bar{\epsilon}\bar{T} + \bar{\partial}(\bar{\mu}\bar{Q}) + \frac{1}{2}\bar{\partial}\bar{\mu}\bar{Q}\,. && (3.7)
\end{aligned}$$

Here we have made the same redefinition (2.27), (2.28) to go from $T', \bar{T}'$ to T, $\bar{T}$.

The functions $T, \overline{T}, Q, \overline{Q}$ can be identified with the 2-d CFT 1-point functions, and they can be derived from the renormalised 3-d super-gravity action written in terms of the sources. The action (up to divergent counterterms which do not change the variation) is, again,

$$W[\mu, \bar{\mu}, \epsilon, \bar{\epsilon}] = I[\Gamma, \bar{\Gamma}] + \frac{k}{4\pi}\int_{\partial M}(R + \frac{1}{2}K[\mu, \bar{\mu}]) \qquad (3.8)$$

whose variation becomes

$$\delta W[\mu, \bar{\mu}, \epsilon, \bar{\epsilon}] = -\frac{1}{2\pi}\int_{\partial M}(T\delta\mu + \bar{T}\delta\bar{\mu} + \delta\epsilon Q + \bar{Q}\delta\bar{\epsilon}) \qquad (3.9)$$

as desired.

3.2 Extended super Virasoro ward identities

We will now obtain the Virasoro Ward identities in the $\mathcal{N}$-extended super-symmetric case. In this case, we work with two copies $\mathrm{Osp}(N|2)$ super-group.

The generalization of (3.2) allows us to impose on both connections the following conditions (we will work here only with the chiral sources μ, ϵ_ρ and set $\bar{\mu} = \bar{\epsilon}_\rho = 0$):

$$\begin{aligned}
\Gamma_x &= \frac{1}{2}\omega J_3 + \frac{1}{k}TJ_+ + J_- + \frac{1}{k}Q_\rho R_+^\rho + E^a B_a && (3.10)\\
\Gamma_x &= \frac{1}{2}\bar{\omega}J_3 + \gamma J_+ + \mu J_- + \psi_\rho R_+^\rho + \epsilon_\rho R_-^\rho + \zeta^a B_a && (3.11)\\
\bar{\Gamma}_z &= \frac{1}{2}\omega J_3 + \delta J_- + \theta_\rho R_-^\rho && (3.12)\\
\bar{\Gamma}_{\bar{z}} &= \frac{1}{2}\bar{\omega}J_3 + J_+ + \xi J_- + \chi_\rho R_-^\rho\,, && (3.13)
\end{aligned}$$

where we have imposed the condition (2.16). This condition does not affect the $\mathrm{SO}(N)$ generators, B_a, which are anti-symmetric matrices.

Solving for a flat connection, we can find the $\mathcal{N}$-extended Super Virasoro-Ward identities:

$$\bar{\partial}T = -\frac{1}{2}\partial^3\mu + \partial(\mu T) + (\partial\mu)T - \eta^{\rho\sigma}(\frac{1}{2}\partial(Q_\rho\epsilon_\sigma) - \partial\epsilon_\rho Q_\sigma) - (\lambda_a)^\sigma{}_\rho E^a\epsilon^\rho Q_\sigma \quad (3.14)$$

$$\bar{\partial} Q_\rho = -\partial^2 \epsilon_\rho + \partial(\mu Q_\rho) + \frac{1}{2}(\partial\mu) Q_m + \epsilon_\rho T + (\lambda_a)_{\sigma\rho}(\partial(\epsilon^\sigma E^a) + E^a \partial \epsilon^\sigma) + \\ + \zeta^a Q^\sigma (\lambda_a)_{\sigma\rho} - \mu(\lambda_a)_{\sigma\rho} E^a Q^\sigma + (\lambda_a)_{\sigma\rho}(\lambda_b)^\sigma{}_\varrho \epsilon^\varrho E^a E^b \tag{3.15}$$

$$\bar{\partial} E^a = \partial \zeta^a + f^a{}_{bc} E^b \zeta^c + \kappa Q^\sigma \epsilon^\rho (\lambda^a)_{\sigma\rho} \,. \tag{3.16}$$

4. Concluding remarks

We have studied in this paper the relationship between supergravity bulk contraints and CFT boundary Ward identities, as predicted by the AdS/CFT correspondence. We have concentrated in the AdS_3/CFT_2 case and have made use of the Chern-Simons formulation for the 3-dimensional bulk supergravity theory. The bulk constraints are then represented by the rather simple equation $F_{z\bar{z}} = 0$ where F is the associated Yang-Mills curvature. The Ward identities are shown to be encoded in this equation, provided the boundary conditions for the gauge field (AdS asymptotics in the Feffermam-Graham sense) are properly incorporated.

We have only considered the saddle point evaluation for the bulk partition function. It would be interesting to go further and compute the full path integral, including also other insertions. Since Chern-Simons theory has no local degrees of freedom this seems feasible. On the other hand, the local group is non-compact, $\mathrm{Osp}(2,1) \times \mathrm{Osp}(2,1)$, and hence the calculation will be challenging.

Finally, it would also be interesting to study the generalization of this approach to the AdS_5/CFT_4 correspondence when the bulk action is taken to be 5-dimensional Chern-Simons supergravity [26]. The holographic energy-momentum tensor for this theory has recently been computed in [27], following the cohomological methods introduced in [28]. The associated Ward identities would probe other aspects of the corresponding dual CFT.

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5. Conventions

The OSp(1|2) group has five generators, three bosonic (which close under SL(2, Re) Lie algebra) and two fermionic ones. We will use the following representation for the generators:

$$J_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad J_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad J_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.1}$$

$$R_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad R_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \tag{5.2}$$

The generators satisfy the following:

$$\mathrm{STr}(J_+J_-) = 1 \qquad \mathrm{STr}(J_3J_3) = \frac{1}{2} \tag{5.3}$$

$$\mathrm{STr}(R_\pm R_\mp) = \pm 1\,. \tag{5.4}$$

The Osp(1|2) group can the generalized to the $\mathcal{N}$-extended super-symmetric case, to get the Osp(N|2, R) group, whose super-Lie algebra is the following:

- The three bosonic generators satisfy (which is just the SL(2, R) Lie algebra):

$$[J_3, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = 2J_3\,. \tag{5.5}$$

- The internal symmetry group generators (which is the SO(N) algebra) satisfy:

$$[B_a, B_b] = f_{abc}B^c\,, \tag{5.6}$$

  where $a, b, c = 0 \ldots N$.

- The fermionic generators satisfy:

$$\{R^\rho_\pm, R^\sigma_\pm\} = \pm\eta^{\rho\sigma}J_\pm \qquad \{R^\rho_\pm, R^\sigma_\mp\} = -\eta^{\rho\sigma}J_3 \pm \kappa(\lambda^a)^{\rho\sigma}B_a\,,, \tag{5.7}$$

  where $\lambda^a = (\lambda^a)^\alpha_\beta$ is the basis of the representation and satisfy the relation $[\lambda^a, \lambda^a] = f^{abc}\lambda_c$, and κ is a normalization constant.

- All the generators are related by:

$$[J_3, R^\rho_\pm] = \pm\frac{1}{2}R^\rho_\pm \qquad [J_\pm, R^\rho_\pm] = 0 \qquad [J_\pm, R^\rho_\mp] = R^\rho_\pm \tag{5.8}$$

$$[B_a, R^\rho_\pm] = -(\lambda_a)^\rho{}_\sigma R^\sigma_\pm \tag{5.9}$$

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